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THE LEGENDRE CONDITION IN OPTIMUM PROBLEMS OF SUPERSONIC GASDYNAMICS

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The necessary Legendre condition for problems of optimum (in the sense of minimum wave drag) supersonic flow past bodies is obtained. Plane and axisymmetric flows are considered on the assumption of imposition of isoperimetric constraints of a general form. Shock-free flows and flows with attached shock waves are investigated. The method here proposed is used for deriving the second order condition in the particular case when it is possible to pass to the reference contour, and which has been earlier obtained by Shmyglevskii [1] and then by Guderley and others [2].

1. Statement of problem. Shock-free supersonic flows (Fig. 1) and with attached shock waves (Fig. 2) past plane and axisymmetric bodies are considered. In Fig. 1 ab is the contour of the body, and ac and bc are the characteristics of the first and second set, respectively. The flow to the left of ac is assumed to be known. In the case of shock-free flow the stream at inlet is generally turbulent. In Fig. 2 ab is the contour of the body, ac is the attached shock wave, bc is the characteristic of the second set, d is the point of contour discontinuity, and dm and dn are characteristics of the first set that bound the rarefaction wave $dmcnd$. The oncoming stream in Fig. 2 is assumed to be uniform and parallel to the x-axis. It is also assumed that inside region abc the flow is supersonic and is free of shock waves.

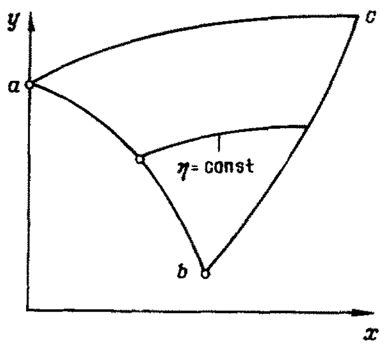

Fig. 1

Velocity projections on the x- and y-axes are denoted by u and v, the pressure by p, the density by ρ, the stream function by ψ with $d\psi = y^\nu \rho\,(u dy - v dx)$, where $\nu = 0$ or $\nu = 1$ in the plane and axisymmetric cases, respectively. Along the body ψ is assumed to be equal unity.

A stationary nonisentropic flow of gas inside the abc-region is defined by equations

$$L_1 = \frac{\partial y^\nu p}{\partial \psi} - \frac{\partial u}{\partial y} = 0, \quad L_2 = \frac{\partial}{\partial \psi}\frac{u}{v} + \frac{\partial}{\partial y}\frac{1}{y^\nu \rho v} = 0 \qquad (1.1)$$

Pressure $p = p\,(u, v, \varphi)$ and density $\rho = \rho\,(u, v, \varphi)$ are defined by relationships

$$\frac{w^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{1}{2}\frac{\varkappa + 1}{\varkappa - 1}, \quad \frac{p}{\rho^\varkappa} = \varphi^{\varkappa - 1}(\psi) \quad (w^2 = u^2 + v^2)$$

where $\varkappa$ is the adiabatic exponent and $\varphi\,(\psi)$ is the entropy function.

Wave drag is defined by the functional

$$\chi = \int_{ab} y^\nu p(x(y), y)\, dy \tag{1.2}$$

The position of point a, and in the case of shock-free flow, also the slope x_a' of contour ab at that point, are assumed to be fixed. One of the coordinates (x_b or y_b) of point b can be arbitrary.

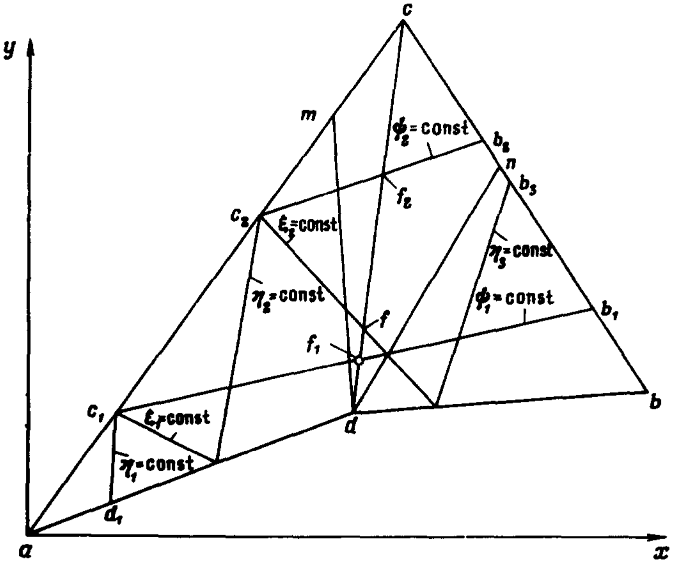

Fig. 2

The isoperimetric condition imposed on the contour of the body is of the form

$$r = \int_{ab} f(x(y), x'(y), y)\, dy \tag{1.3}$$

where $x(y)$ is a function that defines the body contour and x' is its derivative with respect to y.

Along the contour ab the condition

$$vx'(y) - u = 0 \tag{1.4}$$

of no flow through must be satisfied.

We formulate the problem of optimum as follows. Find a function $x(y)$ which yields the minimum of functional (1.2) when Eqs. (1.1) in region abc and conditions (1.3) and (1.4) along the contour ab are satisfied.

In the case of a shock-free flow around a body the condition of stationarity can be satisfied by using a smooth optimum contour. For flows with attached shock waves the situation is different: the optimum contour has generally an infinite number of discontinuities which tend to become denser toward the leading point of the body [3]. To derive the necessary Legendre condition it is sufficient to consider a contour with a single discontinuity (Fig. 2).

2. Shock-free flows. We introduce Lagrange multipliers $\gamma_0(y)$, γ and $h_i(\phi, y)$ and construct the Lagrange functional

$$I = \int_{ab} [y^\nu p + \gamma_0(y)(vx' - u) + \gamma f]\, dy + \iint_S (h_1 L_1 + h_2 L_2)\, d\psi dy$$

We take x' (y) as the control function. If the flow pattern under admissible conditions is to remain unchanged, the quantity $|\delta x'|$ must be fairly small.

The conjugate system for h_i is of the form [4]

$$y^\nu \rho u h_{1\psi} + h_{1y} - \frac{1}{v} h_{2\psi} - \frac{u}{y^\nu \rho v a^2} h_{2y} = 0 \qquad (2.1)$$

$$y^\nu \rho v h_{1\psi} + \frac{u}{v^2} h_{2\psi} - \frac{v^2 - a^2}{y^\nu \rho v^2 a^2} h_{2y} = 0$$

where $a^2 = \varkappa p / \rho$ is the square of the speed of sound.

System (2.1) is supplemented along the body contour by the natural boundary conditions

$$h_1 = -1, \quad h_2 = \gamma_0 v = \gamma \int_{v_b}^{v} \left(f_x - \frac{d}{dy} f_{x'}\right) dy + \gamma^* \qquad (2.2)$$

and along the characteristic bc by

$$h_2 - h_1 y^\nu \rho v^2 \operatorname{tg} \alpha = 0 \quad (\alpha = \arcsin a / w) \qquad (2.3)$$

where α is the Mach angle and γ^* is a constant of integration. We have

$$\delta I = \int_{ab} [y^\nu \delta p + \gamma_0 (\delta v \delta x' + v \delta x' + x' \delta v - \delta u) + \gamma \delta f]\, dy + \qquad (2.4)$$

$$\oint_{abca} \left(y^\nu h_1 \delta p + h_2 \delta \frac{u}{v}\right) dy + \left(h_1 \delta u - h_2 \delta \frac{1}{y^\nu \rho v}\right) d\psi -$$

$$\iint_S \left(y^\nu h_{1\psi} \delta p - h_{1y} \delta u + h_{2\psi} \delta \frac{u}{v} + h_{2y} \delta \frac{1}{y^\nu \rho v}\right) d\psi dy$$

The part of the contour integral (2.4) associated with the characteristic ac vanishes, since the inlet stream is specified and $\delta x_a' = 0$. Along the characteristic bc and contour ab the relationships

$$\frac{dy}{d\psi} = \frac{\sin(\alpha - \theta)}{y^\nu \rho w \sin \alpha} \ (\text{along } bc), \ d\psi = 0 \ (\text{along } ab) \qquad (2.5)$$

where $\theta = \operatorname{arctg} (v / u)$ is the slope of the velocity vector relative to the x-axis, are satisfied.

We represent increments δp, $\delta \rho$, $\delta (1 / \rho v)$, $\delta (u / v)$ and δf in (2.4) in the form of series expansion in powers of δu, δv, δx and $\delta x'$ ($\delta \varphi \equiv 0$, since in the influence region abc (S) shock waves are absent). Having done this, we retain in formula (2.4) linear and quadratic terms, then, using the condition of stationarity and equalities (2.5) after transformation we obtain

$$\delta I = \int_{ab} \gamma \left(\frac{\partial^2 f}{\partial x^2} \delta x^2 + 2 \frac{\partial^2 f}{\partial x \partial x'} \delta x \delta x' + \frac{\partial^2 f}{\partial x'^2} \delta x'^2\right) dy + \qquad (2.6)$$

$$\int_{bc} (a_{11} \delta u^2 + a_{12} \delta u \delta v + a_{22} \delta v^2)\, d\psi + \iint_S (A_{11} \delta u^2 + A_{12} \delta u \delta v + A_{22} \delta v^2)\, d\psi\, dy$$

Analytic formulas for a_{ij} and A_{ij} appear in Sect. 3 below formula (3.1).

The control increments $\delta x'$ are chosen on the assumption that function $\delta x'(y)$ is non-zero only in the interval $[y^\circ, y^\circ + \varepsilon]$ of length ε and that ε is the small parameter of the problem. We define the quantity $\delta x'$ in that interval by another small parameter ε_1. It will be readily seen that the order of magnitude of the increment δx does not exceed $\varepsilon\varepsilon_1$.

To determine the perturbed motion induced by the variation of the contour slope necessitates the consideration of a system of variational equations, which is obtained as the result of linearization of equations of motion relative to the unperturbed flow. Analysis of that system supplemented by linearized boundary conditions shows that outside the characteristic narrow band with base ε the perturbations of stream parameters are of order $\varepsilon\varepsilon_1$, while within the band itself they are of order ε_1.

Let us introduce the characteristic variables ξ and η which will be considered as the independent variables of the problem. The variational equation for the stream function is of the form

$$\delta\psi_{\xi\eta} = b_1(\xi, \eta)\delta\psi + b_2(\xi, \eta), \ \delta\psi_\xi + b_3(\xi, \eta)\delta\psi_\eta \qquad (2.7)$$

$$b_2 = \frac{1}{2b_0}(\Theta_1\xi_x + \Theta_2\xi_y), \quad b_3 = \frac{1}{2b_0}(\Theta_1\eta_x + \Theta_2\eta_y)$$

$$b_0 = \left(1 - \frac{u^2}{a^2}\right)\xi_x\eta_x - \frac{uv}{a^2}(\xi_x\eta_y + \eta_x\xi_y) + \left(1 - \frac{v^2}{a^2}\right)\xi_y\eta_y$$

$$\Theta_1 = \frac{2(1-\varkappa)}{\varkappa}\frac{\varphi'}{\varphi}\psi_x + \frac{2\psi_x\psi_{yy}}{y^{2\nu}\rho^2a^2} - \frac{a_0\psi_x}{y^{2\nu}(w^2 - a^2)}$$

$$\Theta_2 = \frac{1}{y^\nu} + \frac{2(1-\varkappa)}{\varkappa}\frac{\varphi'}{\varphi}\psi_y + \frac{2\psi_y\psi_{xx}}{\varkappa\rho^2a^2} - \frac{a_0\psi_y}{y^{2\nu}(w^2 - a^2)}$$

$$a_0 = (1+\varkappa)\left[y^{2\nu}\frac{\varphi'}{\varphi}\frac{a^2}{\varkappa} + \frac{\psi_{xx}\psi_y^2 + \psi_{yy}\psi_x^2}{y^{2\nu}\rho^2a^2}\right]$$

The specific form of function b_1 is immaterial for the subsequent analysis.

Fig. 3

Let the contour ab be represented in the $\xi\eta$-plane by curve $\xi = \pi(\eta)$ (Fig. 3). As stipulated, function $\delta\psi$ vanishes along the characteristic ac. Its value along the contour ab is determined by the condition

$$\Delta\psi = \delta\psi + \psi_x\delta x = 0$$

where $\Delta\psi$ is the total increment ψ associated with the shift of contour points parallel to the x-axis by the small quantity δx. From the last equality we obtain

$$\delta\psi[\pi(\eta), \eta] = g(\eta)\delta x(\eta), \qquad g(\eta) = -\psi_x(\eta) \qquad (2.8)$$

where η is taken as the independent variable along ab.

Differentiating the expression for $\delta\psi$ along ab, we obtain

$$d\delta\psi / d\eta = \delta\psi_\eta + \pi'(\eta)\delta\psi_\xi = g'(\eta)\delta x(\eta) + g(\eta)\delta x'(\eta) \qquad (2.9)$$

Here and subsequently the prime idicates differentiation with respect to η.

Using equalities (2.7), (2.8) and $\delta\psi|_{ac} = \delta\psi_\xi|_{ac} = 0$, we obtain the following system of integral equations for functions $\delta\psi$ and $\delta\psi_\xi$ (Fig. 3):

$$\delta\psi(\xi,\eta)=\int\limits_{\pi(\eta)}^{\xi}\delta\psi_\xi\,d\xi+O(\varepsilon\varepsilon_1)$$

$$\delta\psi_\xi(\xi,\eta)=b_3(\xi,\eta)\,\delta\psi(\xi,\eta)+\int\limits_0^{\eta}\left[b_2\delta\psi_\xi+\left(b_1-\frac{\partial b_3}{\partial\eta}\right)\delta\psi\right]d\eta$$

whose solution is

$$\delta\psi=O(\varepsilon\varepsilon_1),\quad \delta\psi_\xi=O(\varepsilon\varepsilon_1) \tag{2.10}$$

For the characteristic $\eta=\text{const}$ from (2.7) and (2.10) we obtain

$$\frac{d}{d\xi}\delta\psi_\eta=b_3\delta\psi_\eta+O(\varepsilon\varepsilon_1)$$

By solving this equation with allowance for the boundary condition (2.9) we obtain

$$\delta\psi_\eta(\xi,\eta)=g(\eta)\,\delta x'(\eta)\exp\int\limits_{\pi(\eta)}^{\xi}b_3(\xi,\eta)\,d\xi\equiv R_1(\xi,\eta)\,\delta x'(\eta) \tag{2.11}$$

which is accurate to within quantities of order $\varepsilon\varepsilon_1$. We separate now the principal parts (of order ε_1) of increments δu and δv in formula (2.6) and obtain

$$u=\psi_y/y^\nu\rho,\quad v=-\psi_x/y^\nu\rho,\quad \psi_x=\psi_\xi\xi_x+\psi_\eta\eta_x \tag{2.12}$$

$$\psi_y=\psi_\xi\xi_y+\psi_\eta\eta_y,\quad \frac{\psi_x^2+\psi_y^2}{2y^{2\nu}\rho^2}+\frac{\varkappa}{\varkappa-1}\rho^{\varkappa-1}\varphi^{\varkappa-1}(\psi)=\frac{1}{2}\,\frac{\varkappa+1}{\varkappa-1}$$

By varying these equalities, eliminating in them the variations $\delta\rho$, $\delta\psi_x$ and $\delta\psi_y$, and using formula (2.11), we obtain

$$\delta u=\left[\eta_y-\frac{\psi_y}{\rho}\,\frac{\psi_x\eta_x+\psi_y\eta_y}{y^{2\nu}(\rho w^2-\varkappa p)}\right]\frac{R_1\delta x'}{y^\nu\rho}=k_1(\xi,\eta)\,\delta x' \tag{2.13}$$

$$\delta v=\left[-\eta_x+\frac{\psi_x}{\rho}\,\frac{\psi_x\eta_x+\psi_y\eta_y}{y^{2\nu}(\rho w^2-\varkappa p)}\right]\frac{R_1\delta x'}{y^\nu\rho}=k_2(\xi,\eta)\,\delta x'$$

which is accurate to within quantities of order $\varepsilon\varepsilon_1$.

Passing from variables ψ and y to ξ and η, and using equalities (2.13), we represent the expression for δI as

$$\delta I=\Omega(\eta)\left|\frac{dy(\pi(\eta),\eta)}{d\eta}\right|^{-1}\varepsilon(\delta x')^2+O(\varepsilon^2\varepsilon_1^2) \tag{2.14}$$

$$\Omega(\eta)=\gamma\frac{\partial^2 f}{\partial x'^2}\,\frac{dy(\pi(\eta),\eta)}{d\eta}-\left[(a_{11}k_1^2+a_{12}k_1k_2+a_{22}k_2^2)\,\frac{d\psi(\xi_c,\eta)}{d\eta}\right]+ \tag{2.15}$$

$$\int\limits_{\pi(\eta)}^{\xi_c}(A_{11}k_1^2+A_{12}k_1k_2+A_{22}k_2^2)\,|J|\,d\xi,\quad J=\frac{D(\psi,y)}{D(\xi,\eta)}$$

where the integral is taken along the characteristic $\eta=\text{const}$ of the first set, and the first and second terms are determined at the intersection points of the characteristic with the contour ab and the characteristic bc.

Equation (2.14) yields the necessary condition (the Legendre condition) for the minimum of I

$$\Omega(\eta)\geqslant 0 \tag{2.16}$$

A distinctive feature of this problem is that the control increment along segment ε of the body contour generates inside the narrow band with base ε of the unperturbed stream characteristic an increment of gas velocity of the same order. This is the consequence of the application of the linear theory, which in this case is admissible owing to the smallness of $|\delta x'|$. This is broadly similar to the "variation in a narrow band" applied in problems of control to the principal part of the basic differential operator [5], except that in the latter problems the control is concentrated inside the region, while in the case considered here it operates at the boundary. The appearance in formula (2.15) of terms computed at the characteristics is due to this feature.

If the problem admits passing to the reference contour, condition (2.16) is equivalent to the inequality obtained in [1, 2]. In fact, if we set $f = x'$, then system (2.1) has for h_l the solutions $h_1 = -1$ and $h_2 = \gamma^*$ which satisfy boundary conditions [4, 6], and the problem admits passing to the reference contour. In such case $\partial^2 f / \partial x'^2 \equiv 0$ and the first term in (2.15) vanishes. The integral in (2.15) also vanishes, since A_{ij} are of the form

$$A_{ij} = \sum_{l=1}^{2} \frac{\partial h_l}{\partial \psi} c_l^{ij} + \sum_{l=1}^{2} \frac{\partial h_l}{\partial y} d_l^{ij}$$

Thus in this particular case condition (2.16) reduces to the inequality

$$a_{11}k_1^2 + a_{12}k_1k_2 + a_{22}k_2^2 \geqslant 0$$

or to what is equivalent

$$\operatorname{tg}\theta \geqslant -\frac{\sin 2\alpha\,(1+\cos 2\alpha)}{\varkappa + \cos^2 2\alpha}$$

which is the same as that obtained in [1, 2].

It is important to note that the slope x' of contour ab of the body is the true control. In the particular case when it is possible to formulate the input problem at the reference characteristic bc, it is convenient to consider a specially chosen gasdynamic function defined on bc. For example, the Mach angle α was taken as the control in [1]. The relation between variations $\delta x'$ and $\delta\alpha$ can be readily derived from (2.13) and the relationship $1 + \varkappa = w^2(\varkappa - \cos 2\alpha)$.

3. Flows with attached shock waves. The method of derivation of the necessary Legendre conditions described in Sect. 2 can be used in problems of optimum flow with attached shock waves past a body (Fig. 2). Since the contour of the body has a discontinuity, Eqs. (1.1) must allow in region abc for discontinuous Lagrange multipliers [4]. We assume that the characteristic cd is the line of discontinuous Lagrange multipliers and that in the influence region there are no other such lines. In that case the Lagrange functional is of the form

$$I = \int_{ad} [y^\nu p + \gamma_0^{(1)}(vx' - u) + \gamma f]\, dy + \int_{ab} [y^\nu p + \gamma_0^{(2)}(vx' - u) + \gamma f]\, dy +$$
$$\sum_{i=1}^{2} \iint_{S_i} (h_1^{(i)} L_1 + h_2^{(i)} L_2)\, d\psi\, dy$$

where S_1 and S_2 denote regions abc and adc, respectively.

The conditions of stationarity of I are given in [3]. Unlike (2.6) the formula for increments of I contains variations of the entropy function $\delta\varphi(\psi)$ associated with the change of shape of the shock wave ac. In the considered problem all gasdynamic quantities at points ac depend only on the angle σ of shock wave inclination to the x-axis. Throughout the influence region[1] $\delta\varphi(\psi) = (d\varphi / d\sigma)\delta\sigma(\psi)$, where the derivative $d\varphi \,/\, d\sigma$ is determined at the point of intersection of ac with the streamline $\psi = \text{const}$. The derivation of formula for δI is on the whole similar to the derivation of (2. 6) for a shock-free flow past a body. The result can be presented in the form

$$\delta I = \int\limits_{ab} \gamma \left(\frac{\partial^2 f}{\partial x^2} \delta x^2 + 2 \frac{\partial^2 f}{\partial x\, \partial x'} \delta x \delta x' + \frac{\partial^2 f}{\partial x'^2} \delta x'^2 \right) dy + \tag{3.1}$$

$$\int\limits_{ac} \Phi \delta\sigma^2 \, d\psi + \int\limits_{cb} h_1 (\delta \mathbf{T} a \delta \mathbf{T}) \, d\psi + \int\limits_{cd} \Delta h_1 (\delta \mathbf{T} b \delta \mathbf{T}) \, d\psi +$$

$$\iint\limits_{abc} (\delta \mathbf{T} A \delta \mathbf{T}) \, d\psi \, dy$$

$$2\Phi = \frac{1}{y^\nu w_0} \left(y^\nu h_1 \frac{d^2 p}{d\sigma^2} + h_2 \frac{d^2}{d\sigma^2} \frac{u}{v} \right) + h_1 \frac{d^2 u}{d\sigma^2} - \frac{h_2}{y^\nu} \frac{d^2}{d\sigma^2} \left(\frac{1}{\rho v} \right)$$

where w_0 is the velocity of the oncoming stream; $\delta \mathbf{T}$ is a vector whose components are $(\delta u,\ \delta v,\ \delta\varphi)$; $\Delta h_1 = h_1^{(2)} - h_1^{(1)}$ is the discontinuity of the Lagrange multiplier h_1 on cd ; $a,\ b$ and A are symmetric (3×3)- matrices and

$$A = \frac{n_3}{2\rho^2 v} A_2 - \frac{n_1}{2} A_1 - C$$

$$C_{11} = \frac{n_3}{\rho^3 v} (\rho_u)^2, \quad C_{22} = n_2 \frac{u}{v^2} + \frac{n_3}{\rho v} \left[\frac{1}{v^2} + \rho_v \frac{1}{\rho v} + \frac{1}{\rho^2} (\rho_v)^2 \right]$$

$$C_{33} = \frac{n_3}{\rho^3 v} (\rho_\varphi)^2, \quad C_{12} = - \frac{n_2}{v^2} + \frac{n_3}{\rho v} \left(\frac{1}{\rho v} + \rho_u \frac{1}{\rho v} + \frac{2}{\rho^2} \rho_u \rho_v \right)$$

$$C_{13} = 2 \frac{n_3}{\rho^3 v} \rho_u \rho_\varphi, \quad C_{23} = \frac{n_3}{\rho^2 v} \rho_\varphi \left(\frac{1}{v} + \frac{2}{\rho} \rho_v \right)$$

$$n_1 = y^\nu h_{1\psi}, \quad n_2 = h_{2\psi}, \quad n_3 = h_2\, y y^{-\nu}, \quad \bar{n}_1 = -y^\nu \zeta$$

$$\bar{n}_2 = -y^\nu \rho v^2 \zeta \operatorname{tg} \alpha, \quad \bar{n}_3 = \rho v^2 \operatorname{tg} \alpha$$

$$\zeta = \frac{\sin(\alpha - \theta)}{y^\nu \rho w \sin \alpha}, \quad \bar{\zeta} = \frac{\sin(\alpha + \theta)}{y^\nu \rho w \sin \alpha}$$

(A_1 and A_2 are (3×3)-matrices of second derivatives of functions $p\ (u,\ v,\ \varphi)$ and $\rho\ (u,\ v,\ \varphi)$, respectively).

Formulas for a_{ij} are obtained from A_{ij} by substituting $\bar{n}_i$ for n_i, while those for b_{ij} are obtained from a_{ij} by the substitution of $\bar{\zeta}$ for ζ

In characteristic variables ξ and η region ad^+d^-bc (Fig. 4) corresponds to the influence region; sector d^+d^- corresponds to point d in the physical plane xy and to the extreme position of the characteristic $\xi = \text{const}$ in region $dmcnd$ (Fig. 2).

Let us consider section ad^+ of the contour (Fig. 4). We assume that the variation $\delta x'\ (\eta) = O\ (\varepsilon_1)$ is nonzero only in the interval $[\eta_{d_1},\ \eta_{d_1'}]$ of length ε, and consider ε and ε_1 to be small parameters of the problem (the prime denotes a derivative with respect to η). To determine the perturbation induced by changes of the contour slope it is sufficient to find the relation of variations $\delta\psi$, $\delta\psi_\xi$ and $\delta\psi_\eta$ to $\delta x'$. First, we determine the relation between these variations along the contour ab and along the shock wave ac. Equalities (2. 8) and (2. 9) are satisfied along sections ad^+ and d^-b (Fig. 4), and along section d^+d^- we have

$$\delta\psi(\eta) = g_1(\eta)\, \delta x_d, \quad \delta\psi_\eta = \frac{d}{d\eta} \delta\psi = g_{15}(\eta)\, \delta x_d \tag{3.2}$$

Function $g_1(\eta)$ can be determined by calculating the rarefaction flow in the neighborhood of point d as the $\lim\, [-\psi_x(\eta)]$ for $\xi \to \xi_d$ along the characteristic $\eta = \text{const}$.

At points of the shock wave ac the following functions are known:

$$u = u(\sigma), \quad v = v(\sigma), \quad \rho = \rho(\sigma), \quad \psi = \psi(y)$$

We have

$$\psi_y y^{-\nu} = u(\sigma)\rho(\sigma) \equiv f_1(\sigma), \quad \psi_x y^{-\nu} = -v(\sigma)\rho(\sigma) \equiv f_2(\sigma),$$

$$\psi = \psi(y)$$

By carrying out the complete variation of these relationships, eliminating in them the variations $\delta\sigma$ and δx ($\delta x(y)$ is a small displacement of a point of the shock wave in a direction parallel to the x-axis), and passing to variables ξ and η, we obtain

$$\delta\psi_\xi + a_1\delta\psi_\eta + a_2\delta\psi = 0 \tag{3.3}$$

$$\delta\sigma = a_3\delta\psi_\xi + a_4\delta\psi_\eta + a_5\delta\psi \tag{3.4}$$

$$a_1 = \left(\eta_x \frac{df_1}{d\sigma} + \eta_y \frac{df_2}{d\sigma}\right)\left(\xi_x \frac{df_1}{d\sigma} + \xi_y \frac{df_2}{d\sigma}\right)^{-1}$$

$$a_3 = \xi_y\left(y^\nu \frac{df_1}{d\sigma}\right)^{-1}, \qquad a_4 = \eta_4\left(y^\nu \frac{df_1}{d\sigma}\right)^{-1}$$

The specific form of functions a_2 and a_5 is immaterial in further considerations.

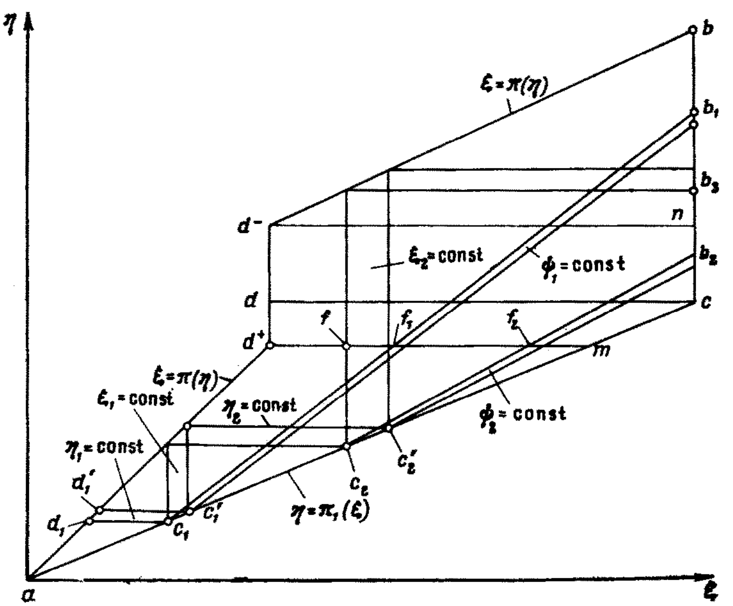

Fig. 4

The analysis of Eq. (2.7) supplemented by boundary conditions (2.8), (2.9), (3.2) and (3.3) shows that the variation $\delta\psi_\eta$ is of order ε_1 only in narrow bands $\eta_i = \text{const}$ with $i = 1, 2, 3$ (Fig. 4), the variation $\delta\psi_\xi$ only in narrow bands $\xi_i = \text{const}$ with $i = 1, 2$. Outside these narrow bands $\delta\psi_\eta = O(\varepsilon\varepsilon_1)$ and $\delta\psi_\xi = O(\varepsilon\varepsilon_1)$. The order of variation of $\delta\psi$ does not exceed $\varepsilon\varepsilon_1$ throughout the influence region.

Analytic formulas for functions $\delta\psi_\xi$ and $\delta\psi_\eta$ within the boundaries of the narrow bands can be readily determined with the use of equalities (2.7), (2.9) and (3.3), since the formula for $\delta\psi_\eta$ in the narrow band $\eta_1 = \text{const}$ (Fig. 4) is provided by (2.11). The expression for $\delta\psi_\xi$ in the narrow band $\xi_1 = \text{const}$ is obtained as the solution of problem $d/(\delta\psi_\xi/d\eta) = b_2\delta\psi_\xi + O(\varepsilon\varepsilon_1)$, $\delta\psi_\xi|_{c_1} = -[a_1\delta\psi_\eta]_{c_1}$.

This means that formula (3.3) represents the law of reflection of perturbations from the shock wave ac, and that of reflection from the contour ab is provided by formula (2.9). Thus perturbations reaching the shock wave ac along the characteristic narrow bands $\eta_i = \text{const}$ are reflected along narrow bands $\xi_i = \text{const}$ in conformity with (3.3), while those reaching the contour ab along the narrow bands $\xi_i = \text{const}$ are reflected along narrow bands $\eta_i = \text{const}$ in conformity with (2.9).

We denote the solutions of related integral equations in the narrow bands $\eta_i = \text{const}$ by $\delta\psi_\eta = R_i\delta x' + O(\varepsilon\varepsilon_1)$, and in the narrow bands $\xi_i = \text{const}$ by $\delta\psi_\xi = P_i\delta x' + O(\varepsilon\varepsilon_1)$. The variation $\delta\sigma$ is of order ε_1 only along segments c_1c_1' and c_2c_2' of the shock wave ac. In accordance with (3.4) we have

$$\delta\sigma = (a_3P_i + a_4R_i)\,\delta x' + O(\varepsilon\varepsilon_1) = c_i\delta x' + O(\varepsilon\varepsilon_1)$$

$$\delta\varphi(\psi) = \frac{d\varphi}{d\sigma}\delta\sigma \equiv d_i\delta x' + O(\varepsilon\varepsilon_1)$$

The quantities δu and δv are obtained by the variation of relationships (2.12)

$$\delta u = t_1\delta\psi_\xi + t_2\delta\psi_\eta + t_3\delta\varphi, \quad \delta v = t_4\delta\psi_\xi + t_5\delta\psi_\eta + t_6\delta\varphi$$

$$t_3 = -\frac{\psi_y a^2}{y^\nu\varphi\,(\rho w^2 - \varkappa p)}, \quad t_6 = \frac{\psi_x a^2}{y^\nu\varphi\,(\rho w^2 - \varkappa p)}, \quad t_2 = \frac{k_1}{R_1}, \quad t_5 = \frac{k_2}{R_1}$$

(see (2.13)); formulas for t_1 and t_4 are obtained from t_2 and t_5 by substituting derivatives ξ_x and ξ_y for η_x and η_y

To obtain the Legendre condition it is necessary to know the transverse dimensions of the narrow bands. Let $\Delta\eta_i$, $\Delta\xi_i$ and $\Delta\psi_i$ be the dimensions of bands. We have

$$\Delta\eta_1 = \varepsilon \equiv r_1\varepsilon, \quad \Delta\xi_1 = \frac{\varepsilon}{\pi_1'(\xi)\,|_{c_1}} \equiv l_1\varepsilon, \quad \Delta\psi_1 = \frac{d\psi\,[\xi, \pi_1(\xi)]}{d\xi}\bigg|_{c_1} l_1\varepsilon \equiv m_1\varepsilon$$

where $\eta = \pi_1(\xi)$ is the equation of the shock wave ac.

We obtain in the same way $\Delta\eta_i = r_i\varepsilon$, $\Delta\xi_i = l_i\varepsilon$ and $\Delta\psi_i = m_i\varepsilon$.

Formula (3.1) for δI can now be written in the form (see Figs. 2 and 4)

$$\delta I = \Omega(\eta_{d_1})\varepsilon\,(\delta x')^2 + O(\varepsilon^2\varepsilon_1^2) \tag{3.5}$$

$$\Omega(\eta_{d_1}) = \left[\Upsilon\frac{\partial^2 f}{\partial x'^2}\frac{dy}{d\eta}\right]_{d_1} + \sum_{i=1}^{2}\Phi|_{c_i} m_i c_i^2 - \sum_{i=1}^{2} m_i\,[(h_1\mathbf{q}_i^3 a\mathbf{q}_i^3)|_{b_i} + \tag{3.6}$$

$$(\Delta h_1\mathbf{q}_i^3 b\mathbf{q}_i^3)|_{f_i}] + r_3\left[h_1(\mathbf{q}_3^1 a\mathbf{q}_3^1)\frac{d\psi}{d\eta}\right]_{b_3} - l_2\left[\Delta h_1(\mathbf{q}_2^2 b\mathbf{q}_2^2)\frac{d\psi}{d\xi}\right]_f +$$

$$\sum_{i=1}^{3} r_i \int\limits_{\eta_i=\text{const}} [(\mathbf{q}_i^1 A\mathbf{q}_i^1)\,|J|]\,d\xi + \sum_{i=1}^{2} l_i \int\limits_{\xi_i=\text{const}} [(\mathbf{q}_i^2 A\mathbf{q}_i^2)\,|J|]\,d\eta +$$

$$\sum_{i=1}^{2} |m_i| \int\limits_{\psi_i=\text{const}} (\mathbf{q}_i^3 A\mathbf{q}_i^3)\,dy$$

$$q_i^1 = (t_2R_i,\ t_5R_i,\ 0),\ q_i^2 = (t_1P_i,\ t_4P_i,\ 0),\ q_i^3 = (t_3d_i,\ t_6d_i,\ d_i)$$

where the derivative $dy\,/\,d\eta$ in the first term is taken along the contour ad^+ and the derivatives $d\psi/d\eta$ and $d\psi/d\xi$ in the fourth and fifth terms are taken along the characteristics bc and cd, respectively.

The necessary condition for minimum I (the Legendre condition)

$$\Omega\ (\eta_{d_1}) \geqslant 0 \tag{3.7}$$

follows from (3.5).

The terms in formula (3.6) (except the first) for Ω can be divided into three groups whose origin is associated with perturbation propagation in the narrow bands along the characteristics η_i = const, ξ_i = const, and the streamlines ψ_i = const. The first, second and third groups contain multipliers r_i, l_i and m_i, respectively, and each of these groups contains integral terms and terms outside the integrals. The presence of integral terms is due to perturbations propagating along related narrow bands, while terms outside the integrals appear as the consequence of intersection of these bands with the characteristics bc and cd and with the shock wave ac.

Note that the number n of perturbation reflections from the shock wave (in Fig. 2 $n = 2$) depends on the position of point d_1 on the contour section ad^+. The above analysis is readily applicable to any arbitrary n.

If the initial point d_1 is located in section d^-b of the contour ab, the perturbation wave that propagates from the point along the characteristic η = const, reaches the characteristic bc and continues beyond the boundaries of the influence region abc. In that case formula (3.6) for Ω contains an integral along the characteristic η = const and two integrated terms which are computed at the intersection points of that characteristic with the contour d^-b and with the closing characteristic bc. This case is analogous to that considered in Sect. 2 of shock-free flow around a body. Owing to this, the inequality (3.7) coincides with (2.16) as regards points of contour d^-b.

At the discontinuity point d of contour ab the following two limit inequalities must be satisfied:

$$\Omega\ (\eta_{d+}) \geqslant 0,\quad \Omega\ (\eta_{d-}) \geqslant 0$$

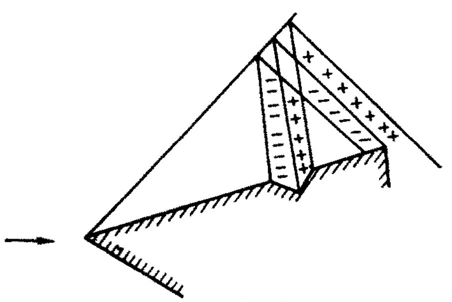

Fig. 5

The method of slope variation along a small section of the body surface was used by Chernyi [7] in the case of attached shock wave generation for proving that for a specified ratio of body thickness to its length the wedge is not a body of minimum wave drag. He varied the initial contour as shown in Fig. 5 (regions of increased and diminished pressure are denoted by plus and minus signs, respectively). The wave drag reduction is explained by that the rarefaction wave reflected from the shock wave without change of sign (for a positive reflection coefficient) lowers the pressure at the right-hand end of the body, while the compression wave reflected from the shock wave does not reach the body. This method was used in [8, 9] in the problem of finding the optimum shape of a body in the presence of a tangential discontinuity and, also, in the problem of the composite nozzle.

The above considered negative decompensation of pressure perturbations in the first instance along the contour of the body is eliminated by the introduction of a contour discontinuity [3, 8, 9]. This result was obtained in investigations of the properties of the first variation of the minimizing functional. The problem of negative decompensation elimination in the second order can be solved by analyzing the necessary conditions of the Legendre kind.

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ON THE BEHAVIOR OF SOLUTIONS OF EQUATIONS FOR DOUBLE WAVES IN THE NEIGHBORHOOD OF THE QUIESCENT REGION

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The structure of solutions of gasdynamic equations is investigated in the case of unsteady double waves in the neighborhood of the quiescent region. A general concept of double waves is presented in the form of special series with logarithmic terms. Results of numerical computations are given.